FINAL
12/7/92
Physics 114B

1. (10 points)(a) Find the general solution to the equation

$$\frac{d^2y}{dx^2} + \frac{\omega^2}{x^2}y = 0$$

where $\omega > 1/2$ and is a constant. *Hint: Try a power law.*

(b) What is the solution when $y(1) = 0$? Use the fact that $x^{i\alpha} = \exp(i\alpha \ln x)$ and $\sin(\theta) = (\exp(i\theta) - \exp(-i\theta))/2i$, to express your result in terms of sine, logarithm, and $\sqrt{\ }$.

2. (30 points) Consider the problem of a string with a line density that is proportional to $1/x^2$ along its length. It obeys the equation

$$\frac{1}{x^2}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}$$

y(x,t) is the vertical displacement as a function of time. In this problem $1 < x < 2$ and the the string is tied at the ends $y(x = 1, t) = y(x = 2, t) = 0$. Find the frequencies of vibration as follows.

(a) Separate variables, using $y(x, t) = X(x)T(t)$. The equation you get for T should be $d^2T/dt^2 + \omega^2 T = 0$.

(b) Solve the equations for $T(t)$ and $X(x)$, and apply the boundary condition $y(x = 1, t) = 0$. *Hint: see problem 1.*

(c) Apply the boundary condition $y(x = 2, t) = 0$, to determine what values of ω are possible.

(d) Write down the general solution for $y(x, t)$ as a sum over the solutions $X(x)T(t)$.

(e) Draw a rough sketch of the lowest frequency mode, and also a sketch of a high frequency mode.

3. (10 points) An external force is applied to the string of problem 2 leading to the equation

$$\frac{1}{x^2}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2} + F(x, t)$$

$F(x, t)$ is varies sinusoidally in time as $F(x, t) = -\exp(i\omega t)f(x)$. By assuming $y(x, t) = \exp(i\omega t)X(x)$, find the differential equation for $X(x)$. *ω is arbitrary and does not have to be a frequency of vibration of the free string that was obtained in 2(c).*

4. (30 points) The equation obtained in problem 3 can be solved in general using Green's functions, by going through the following steps.

(a) First consider the case $f(x) = \delta(x-x')$. In this case $X(x) = G(x, x')$. Find the solution to the equation obtained in the last problem for $x < x'$ by applying the boundary condition $X(1) = 0$ to the general solution of problem 1(a).

(b) Find the general solution for $x > x'$ by applying the boundary condition $X(2) = 0$ to the general solution of problem 1(a).

G(x,x') is therefore of the form

$$G(x, x') = \begin{cases} A(x')X_1(x) & \text{for } x < x' \,, \\ B(x')X_2(x) & \text{for } x > x' \,, \end{cases}$$

where X_1 and X_2 were found in parts (a) and (b).

(c) By integrating the equation obtained in problem 3 over x from $x' - \epsilon$ to $x' + \epsilon$, obtain the the change in the derivative of G from $x < x'$ to $x > x'$.

(d) Using (b) and the fact that G is continuous, solve for $A(x')$ and $B(x')$. The answer is greatly simplified by the identity $\sin(a - b) = \sin a \cos b - \cos a \sin b$.

(e) Now that the Green's function has been obtained, what is the general solution to equation obtained in problem 3, for any $f(x)$? (You don't have to write out the whole answer explicitly, just enough to make it clear that you know what you're doing).

5. (20 points) (a) How many different combinations of 5 cards out of a deck are there?

(b) What is the probability of obtaining a Royal Flush? (A Royal Flush is Ace, King, Queen, Jack, and 10, all of the same suite.)