Read Gould and Tobochnik Thermal and Statistical Physics Chapter 5. Look at the simulation applet problems, they’re very instructive.

Do problems:

Sethna 9.5 (see Chapter 9 section 9.5 of Gould and Tobochnik for a treatment of many aspects of this problem),

Reif 10.1,

Gould and Tobochnik 5.22. This is most easily accomplished by employing the canonical ensemble and converting a term similar to \( \exp(M^2/2) \) to a term proportional to 
\[
\int_{-\infty}^{\infty} e^{\lambda M} e^{-\lambda^2/2} d\lambda
\]

1. Consider the one dimensional Ising model for \( N \) spins. The Hamiltonian

\[
H = -J \sum_{i=1}^{N-1} s_is_{i+1}, \quad s_i = \pm 1
\]

(a) Find the formula for the correlation function \( \langle s_is_j \rangle \), as a function of \( i - j \).

\textit{Hint: Use the change of variables} \( \sigma_i = s_is_{i+1} \). \textit{You should end up with a Hamiltonian in terms of these variables and possibly} \( s_1 \), \textit{and find that they are all non-interacting}.

(b) How does the correlation length in the above formula depend on temperature? How does the correlation function depend on the sign of \( J \)?

2. Consider the one dimensional Ising model Hamiltonian

\[
H = -J \sum_{i=1}^{N} s_is_{i+1}, \quad s_i = \pm 1
\]

(a) What is \( \langle s_i \rangle \)?
(b) The Hamiltonian is now changed, by adding an extra term acting only on the first spin $H_1 = -hs_1$. Now what is $\langle s_i \rangle$? See hint to problem 1(a)

(c) How is $\langle s_i \rangle$ related to the correlation function $\langle s_1 s_i \rangle$?

3. $N$ Ising spins $S_i = \pm 1, i = 1 \ldots N$ are all connected to a central spin $S_0$, but not each other, through a ferromagnetic coupling $J$ as shown below.

They are all in a uniform magnetic field. The Hamiltonian for the system is

$$H = -J \sum_{i=1}^{N} S_0 S_i - h \sum_{i=0}^{N} S_i$$

(a) Calculate the partition function for arbitrary $N$.

(b) For very large $N$ calculate $\langle S_0 \rangle$ as a function of $\beta$ and $h$.

(c) In the large $N$ limit, state if there is a discontinuity in $\langle S_0 \rangle$ as a function of $h$, and for what temperature(s) it appears. Calculate the size of the discontinuity.

(d) In the large $N$ limit, calculate $\langle S_i \rangle$ for all $i \leq N$, as a function of $\beta$ and $h$.

4. A system of spins is described by the vector model

$$H = -\frac{J}{2} \sum_{i,j} s_i \cdot s_j$$

where the $s_i$’s are unit vectors living in two dimensions and $i, j$ are nearest neighbors on a two dimensional square lattice of dimensions $L \times L$. Take $J > 0$ and ignore quantum effects.
(a) What is the ground state and ground state energy?

(b) Find an approximate expression for $H$ at low temperatures by writing the spin variables in terms of the angular deviation from the ground state $\theta_i$. Your answer should involve terms no higher than second order in the $\theta_i$'s.

(c) Using this write down the partition function at low temperatures. What is the specific heat?

5. Consider the Hamiltonian

$$H = -J \sum_{i=1}^{N} S_i S_{i+1} - h \sum_{i=0}^{N} S_i$$

but now where $S_i$ can take on the values $-1, 0, \text{ and } 1$. Calculate the free energy for large $N$ in this case. *Hint: read Gould and Tobochnik 5.4.4.*